Evaluation of the statistical parameters of a Weibull distribution

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A simple iterative procedure for determination of the statistical parameters of a Weibull distribution is proposed. All experimental results on specimens of different size are considered together as a statistically representative population. The procedure can be used for a population in which each specimen has a unique size. The statistical reliability of the iterative procedure is illustrated by comparison with a minimization analysis and confirmation with existing methods. Experimental confirmation of the analysis is developed using six types of glass and carbon fibres at four gauge lengths each. It is shown that Weibull parameters, obtained separately for populations of fixed length, vary with the fibre length.

1. Introduction

The use of Weibull statistics [1] to characterize the failure of brittle materials is well established. Moreover, a probabilistic model utilizing a weakest link concept allows one to predict the dependence of strength on material (specimen) size. In accordance with the classical Weibull size effect, one should expect that statistical information regarding the strength distribution at material volume of V_0, for example, may be used directly to calculate the distribution at volume, V. However, independent experimental results on different specimen sizes of the same materials often show significant variability of the distributions and, especially, shape parameters. This well-known phenomenon brings into question the reliable extrapolation of a Weibull distribution obtained at one volume to obtain a distribution for a population of different size. We assume that the statistical analysis of the size effect is best carried out using a single large population of specimens of various sizes. Such an approach will permit one to obtain the most plausible estimations of Weibull parameters within the domain of considered sizes and be a clearer indication as to whether a Weibull distribution will properly characterize the size effect with a single set of statistical parameters.

The following existing methods for the above-mentioned statistical problem may be noted: (a) a method of average values; (b) a least square method (LSM); (c) a method of maximum likelihood. Method (a) consists of two main steps, namely calculation of the average strengths for each size, followed by a linear approximation of the dependence of the average strength on the size. This widely used approach has serious disadvantages. Firstly, instead of using all experimental points together to calculate the statistical parameters, only the average characteristics of a limited number of sizes are considered. (In practice, rarely more than four different sizes are used for the analysis.) Secondly, in general, each average strength is calculated from a different number of measurements and the non-uniform statistical "weight" of each average value is neglected. These disadvantages are reflected in low statistical reliability of the results. While method (b) takes into account all experimental points, it is assumed that there are large enough populations for each size [2]. Even populations of five to ten specimens for each size may be too small while using LSM. In a case when each specimen has a unique size, this method may provide too rough (and perhaps, incorrect) estimations. Experimental analysis of a multi-step failure is an example of experimental programs in which each following failure reflects a unique size of the remaining virgin part of a specimen. Fibre fragmentation in a single-fibre composite is a typical application of a multi-step experimental programme [3]. Thus, use of the LSM technique can result in an unreliable estimate of the size effect. Strictly speaking, method (c) has none of these disadvantages [4]. However, its application is reduced to a problem of a non-linear function minimization [5], in which numerical realization may be accompanied by certain difficulties.

Thus, the purpose of the present work was to develop a simple approach and respective statistical procedure for predicting Weibull parameters using all experimental data obtained at various specimen sizes. Glass and carbon fibres are brittle in nature and their length may be considered as a characteristic size parameter. Consequently, experimental confirmation of the approach and verification of the suitability of a Weibull distribution for describing the size effect has been carried out on six different sets of the fibres, each having data for four different gauge lengths.

2. Theory

Let us consider a strength distribution of a brittle material with volume, V in a classical Weibull form [1]

$$P(\sigma, V) = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{A}\right)^{\beta}\right] \quad (1)$$

where A, β is the scale, shape parameter and V_0 is a reference volume. The volume V may be represented as kV_0, and thus, Equation 1 has a form

$$P(\sigma, V) = 1 - \exp\left[-k\left(\frac{\sigma}{A}\right)^{\beta}\right] = 1 - \exp\left[-\left(\frac{s}{A}\right)^{\beta}\right] \quad (2)$$

where

$$s = \sigma k^{1/\beta} \quad (3)$$

is the reduced stress calculated for each considered volume. The statistical parameters A and β from Equation 2 may be easily calculated using a traditional LSM for a plot $y_i \rightleftharpoons x_i$; $i = 1, \ldots, n$, in a linear form

$$y = \beta x + \alpha \quad (4)$$

where

$$x_i = \ln(s_i) \quad (5a)$$

$$y_i = \ln\{\ln[1/(1 - P_i)]\} \quad (5b)$$

$$P_i = (i - 0.5)/n \quad (5c)$$

$$\alpha = -\beta\ln(A) \quad (5d)$$

and n is the number of tests. The only complexity is connected with initial evaluation of s_i, because its calculation requires the use of an unknown value of β. Therefore, the following iterative approach is proposed.

(a) An initial value of β is introduced. (As we will see below, the choice of the initial value does not affect the final result.)

(b) Values of s_i, $i = 1, \ldots, n$ are calculated taking into account the size (volume) of each specimen.

(c) Parameters A and β' are calculated using the LSM in the conventional manner.

(d) If the difference $|\beta - \beta'|$ is greater than a requested accuracy, the iterative process is continued by returning to the step b at $\beta = \beta'$. Otherwise, the convergence process is assumed to be completed.

3. Experimental procedure

Experimental confirmation of the technique was carried out using tensile strength data from the testing of the six types of fibres listed in Table I. The fibre diameters were measured using an optical microscope (ORTHOLUX II POL-BK by Leitz) and an image analyser system (MET1 by Pertel) at a magnification of $\times 400$. From observations performed with a scanning electron microscope (Stereoscan 200 by Cambridge), we can say that all the fibres appear to have an almost circular cross-section.

Tensile strengths were measured at various gauge lengths on monofilaments randomly extracted from a bundle of the fibre to be tested. Four different gauge lengths have been considered for each type of fibre: $l_1 = 20$ mm; $l_2 = 15$ mm; $l_3 = 10$ mm; $l_4 = 5$ mm, except for: $l_1 = 40$ mm; $l_2 = 20$ mm for fibre c and $l_1 = 25$ mm; $l_2 = 20$ mm for fibre e. In accordance with ASTM standard D3379-75 [6], a single fibre was centre-line mounted on special slotted thin paper tabs using a quick-setting glue. The tabs were gripped so that the test specimen was aligned axially in the jaws of a constant-speed movable-crosshead test machine. The paper of the mounting tabs was then cut away. Tests were conducted at room temperature and at a constant crosshead speed of 0.2 mm min^{-1} using an Instron 4502 tensile tester equipped with a 10 N load cell. All the fibres exhibit brittle failure and a linear force–deflection response to the point of failure.

Results of statistical treatment for each gauge length separately are presented in Table II where $\bar{R}$ is the average strength and v is the coefficient of variation. Experimental values of the average strength decrease with increasing fibre length, indicating a significant size effect for each type of fibre. Moreover, considerable variability, v up to 35%, reflects the evident stochastic nature of the fibre failure. The Weibull parameters presented in Table II should not be functions of the fibre length (or volume), if Equation 1 is an appropriate form for the various distributions. However, values of the scale parameters presented in Table II (reduced to $l_4 = 5$ mm: $A_{li} = A(l_i/l_4)^{1/\beta_i}$) and the shape parameters β vary significantly with fibre length.

4. Discussion

Results of statistical treatment using the proposed iterative procedure are shown in Table III at a reference length of $l_o = l_4 = 5$ mm. The convergence process is fast, and exactness of three digits for β is obtained at three to four iterations. Moreover, the

TABLE I Description of fibres

Fibre	Manufacturer	Trade name	Material	Surface treatment	Diameter (μm)
a	Sisecam	–	E–glass	Bare	14.8 ± 1.3
b	Sisecam	–	E–glass	PA compatible	14.7 ± 1.1
c	PPG	2001	E–glass	epoxy compatible	25.1 ± 0.4
d	Vetrotex	P375	E–glass	PA compatible	19.9 ± 0.9
e	Vetrotex	P5213	E–glass	PP compatible	16.8 ± 1.2
f	Toho	Besfight	carbon	epoxy compatible	7.0 ± 0.1

TABLE II Statistical characteristics of strength depending on fibre length

Charact.	l	Fibre a	b	c	d	e	f
n	l_1	27	23	23	26	17	21
	l_2	31	22	23	27	17	27
	l_3	24	23	20	32	22	18
	l_4	29	34	18	30	19	16
	Σ	111	102	84	115	75	82
A (MPa)	l_1	2020	2565	4218	2602	2146	3764
	l_2	1793	2558	2341	2703	3220	3967
	l_3	1768	2640	2207	2472	3364	3945
	l_4	2017	2628	2184	2400	2777	3765
β	l_1	3.18	7.07	2.76	7.94	4.44	5.72
	l_2	5.13	7.76	6.47	6.50	3.67	5.80
	l_3	6.61	6.85	8.14	9.38	3.17	6.00
	l_4	4.16	5.52	5.65	8.74	3.44	9.81
$\bar{R}$ (MPa)	l_1	1168	1968	1730	2058	1363	2734
	l_2	1331	2088	1760	2125	1989	3041
	l_3	1485	2230	1911	2178	2409	3259
	l_4	1823	2426	2019	2271	2488	3578
Δ (%)	l_1	− 7.36	− 1.12	14.33	2.62	− 22.89	− 4.75
	l_2	− 0.68	0.62	1.08	2.35	10.31	1.54
	l_3	1.08	1.26	− 6.96	− 0.13	9.92	2.18
	l_4	5.65	− 0.37	− 18.97	− 4.53	− 5.87	0.84
ν (%)	l_1	35.4	16.3	31.0	15.1	25.6	20.9
	l_2	23.3	14.3	18.0	17.1	30.2	20.6
	l_3	17.7	17.2	14.6	12.9	32.7	19.4
	l_4	23.5	21.5	20.4	13.6	34.9	11.8

TABLE III Statistical parameters of Weibull distribution at the reference length $l_0 = 5$ mm

Fibre	n	A (MPa)			β			r		
		Iterative[a]	LSM[a]	MAV[a]	Iterative[a]	LSM[a]	MAV[a]	Iterative[a]	LSM[a]	MAV[a]
a	111	1888	1975	2048	4.39	3.82	3.20	0.9968	0.9441	− 0.9959
b	102	2606	2631	2619	6.87	6.50	6.74	0.9823	0.9479	− 0.9928
c	84	2639	2601	2086	4.32	4.46	12.70	0.9818	0.8682	− 0.9742
d	115	2518	2513	2360	8.18	8.23	14.60	0.9918	0.9593	− 0.9926
e	75	2931	3071	3018	3.54	3.15	3.12	0.9851	0.9268	− 0.8446
f	82	3809	3897	3938	6.47	5.82	5.44	0.9775	0.9444	− 0.9732

[a] "Iterative" are the results calculated by the proposed approach, "LSM" by the least square method, "MAV" by the method of average values.

convergence of A is developed even faster. The initial value of β was arbitrarily chosen as 1, although the same final values were obtained using an initial value of $\beta = 100$. (We recommend, however, that the initial value be chosen close to an "expected" one.) Graphical interpretation of the distributions is presented in Fig. 1. The linear character of $\ln\{\ln[1/(1-P)]\} \leftrightarrow \ln(s)$ and the magnitudes of the coefficients of correlation, r, confirm the linearity of these dependencies (Table III).

The approach permits one to solve an indirect problem, and predicts regularities of the size effect as well. In other words, the average strength at an arbitrary length, l, may be calculated as

$$\bar{R}_{\text{theor}} = A\Gamma(1 + 1/\beta)(l/l_o)^{-1/\beta} \qquad (6)$$

Table II shows the difference

$$\Delta = \frac{\bar{R} - \bar{R}_{\text{theor}}}{\bar{R}} \qquad (7)$$

between the predicted value of $\bar{R}_{\text{theor}}$ using Equation 6 and the value of $\bar{R}$ from experimental results. One can note that with the exceptions of fibres c at l_1, l_4 and e at l_1, the differences are relatively small. Therefore, the Weibull parameters obtained using the iterative calculation, utilizing data on all fibre lengths, accounts properly for the effect of fibre length on the strength.

A mathematical confirmation of the approach may be attained by comparison with a minimization procedure. In contrast with the iterative technique, the following alternative method of solution may be used as well. Again, we assume that the initial value of β is known. The LSM for the conventional relation expressed by Equation 4 may be written as

$$\Phi = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta x_i - \alpha)^2 \rightarrow \min \qquad (8)$$

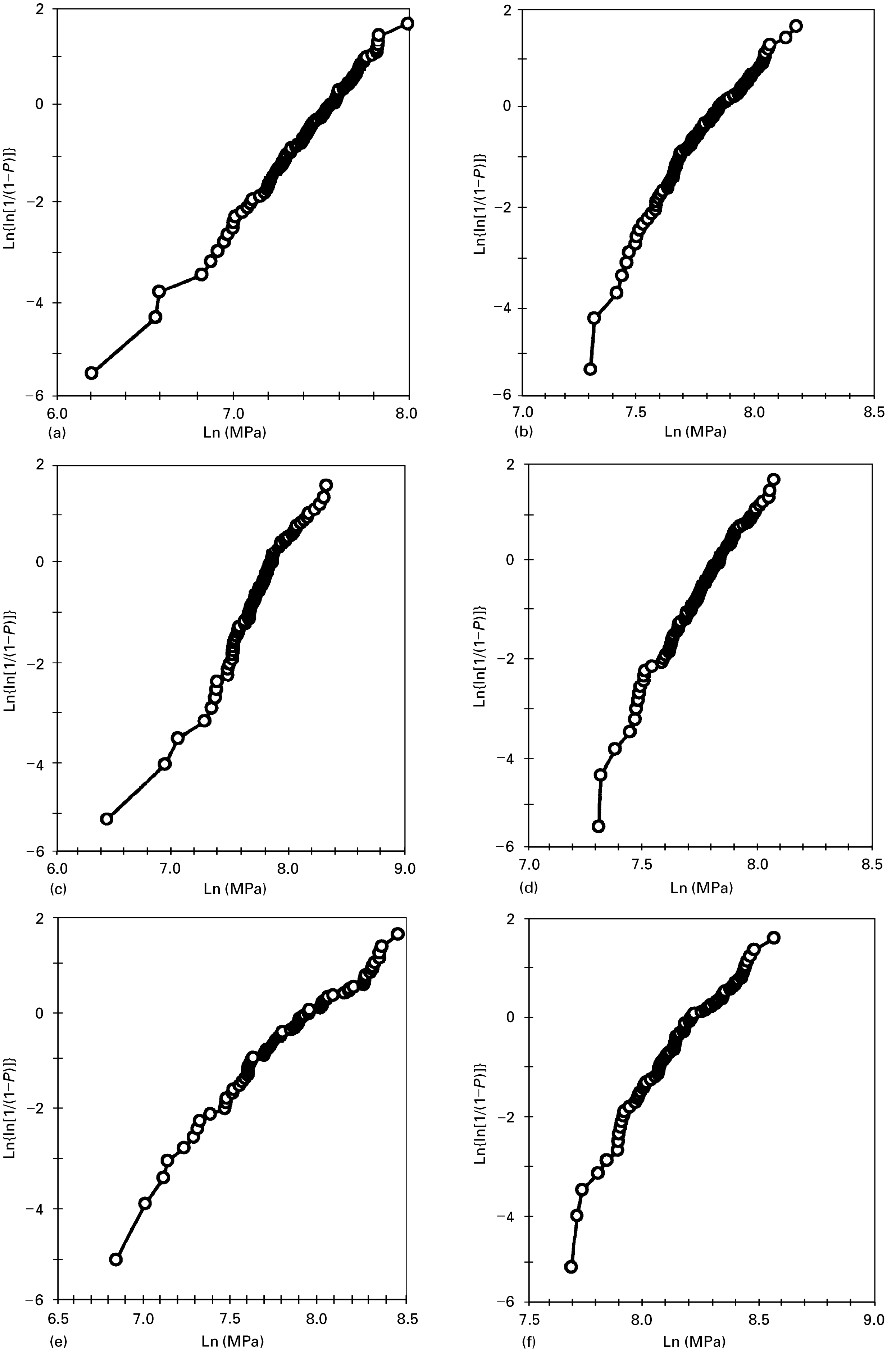

*Figure 1(*a–f) Cumulative probability functions of the reduced fibre strength.

Solving the problem for α at $\partial\Phi/\partial\alpha = 0$, one obtains

$$\alpha_{\min} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta x_i)^2 \qquad (9)$$

Therefore, Equation 8 can be presented as an one-dimensional convex problem

$$\Phi = \Phi[\beta, \alpha_{\min}(\beta)] \rightarrow \min \qquad (10)$$

which can be solved numerically. For example, the approach represented by Equations 8–10 results in values of $\beta = 4.385$; $A = 1888.8$ MPa for fibre *a* (exactness 0.001 of β); while the iterative procedure produces $\beta = 4.389$, $A = 1888.0$ MPa, i.e. the same values for the statistical parameters. This example shows the correctness of the proposed iterative approach.

Let us further consider a difference between the proposed iterative approach and the LSM. Evaluations of the parameters including relevant coefficients of correlation are presented in Table III, while a graphical representation of the LSM is shown in Fig. 2 as a plot $\ln\{\ln[1/(1-P)]\} - \ln(k) \rightleftharpoons \ln(\sigma)$. Although the iterative approach and LSM provide comparable results, there are certain inevitable differences between them (Table II). The differences may be explained by the approximate nature of the magnitudes P_i calculated using Equation 5 as $P_i \approx (i-0.5)/n$. Using the proposed approach, all experimental data are considered together and $n = \sum n_j$, where n_j is the number of points for *j*th length. Using the LSM, the probabilities $P_{ij} \approx (i-0.5)/n_j$ are considered separately for each length. Because n is always more than n_j, one can assume that the iterative approach provides more exact evaluations. While an increase in the number of data points used will reduce the differences, use of a finite number of the experimental tests will always be reflected by the differences in the calculated values using the two methods.

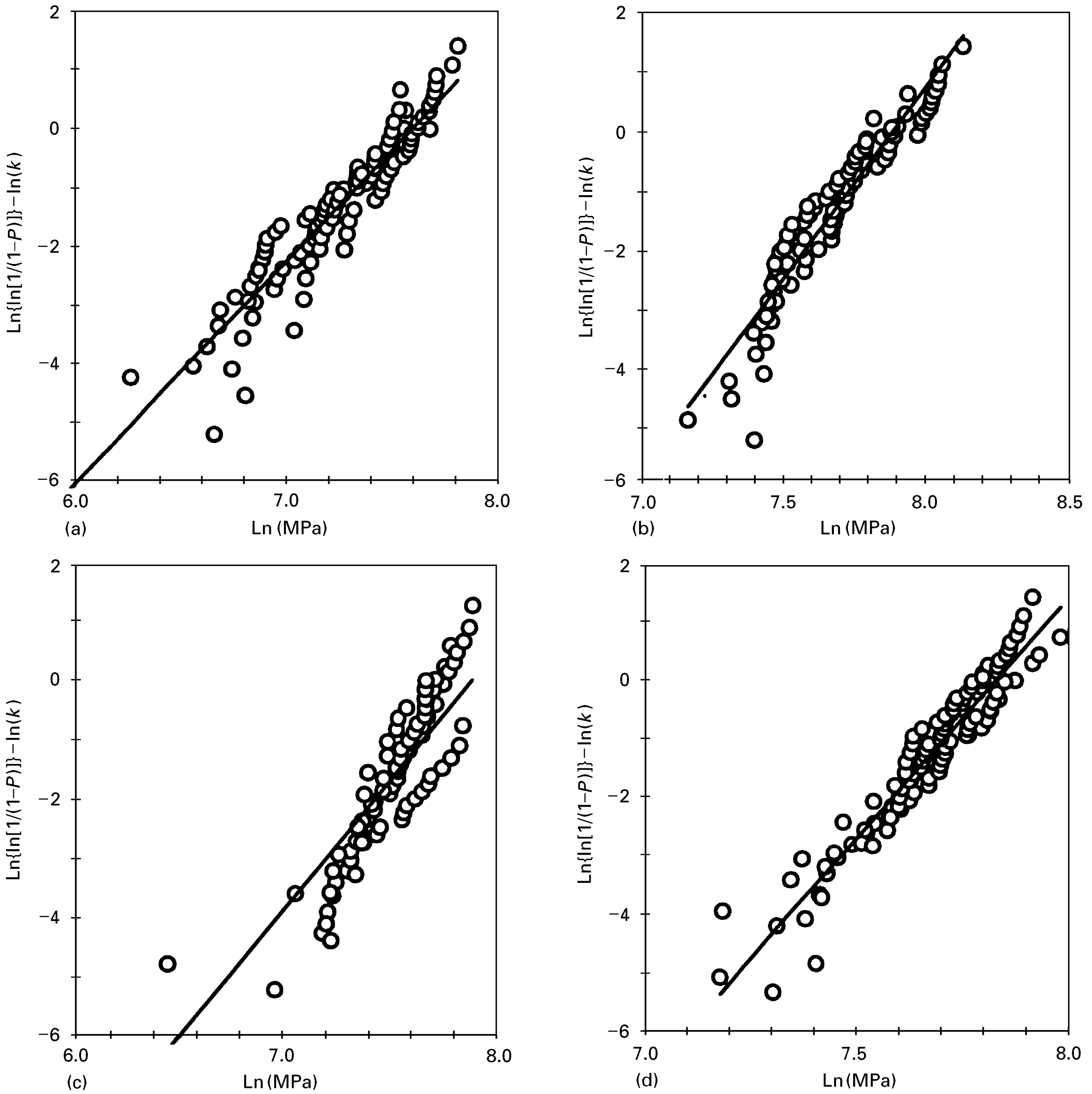

Figure 2(a–f) Graphical representation of the fibre strength distribution calculated using the LSM for fibre types a–f, respectively.

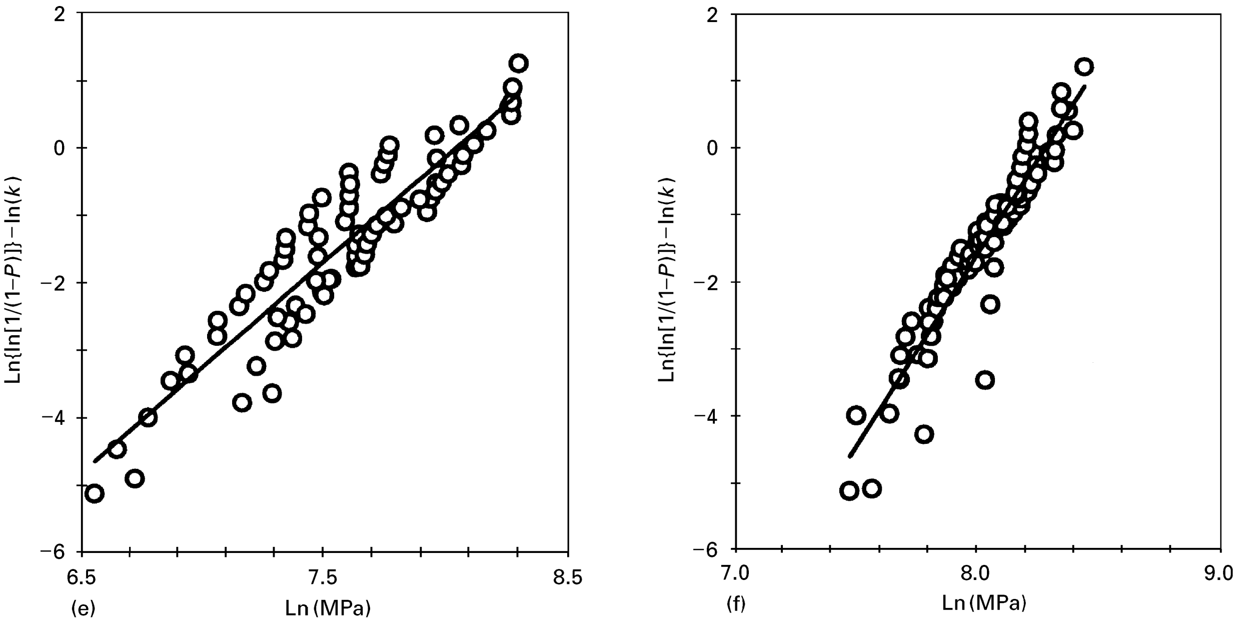

Figure 2 (Continued)

One can note that application of the method of average values may be associated with serious overestimation (Table III). For example, parameters β and *A* for fibres *c* and *d* calculated using this method are too far from those evaluated by the iterative approach or LSM. The lowest statistical reliability of this method is explained by a small number of analysed points, which are determined by the number of different lengths.

5. Conclusion

1. The primary advantage of the proposed iterative approach allows one to utilize all the experimental data on the strengths of various specimen sizes to obtain a single set of statistical parameters which properly accounts for the size effect on the strength distribution. Utilization of this larger-sized, more diverse population reflects all experimental data, and therefore, provides a more reliable prediction of the strength distribution within the domain of considered sizes.

2. The proposed iterative strategy provides a simple numerical procedure for calculation of the statistical parameters using ordinary software of the LSM. The convergence process is fast and provides correct results, as confirmed by comparison with a direct minimization procedure.

3. When applied on the experimental data for six different fibres, each at four lengths, the proposed statistical treatment results in a single set of statistical parameters for each fibre. The linear dependence of the reduced stresses, *s* shown in Fig. 1, and the coefficients of correlation approaching unity reported in Table III, attest to the suitability of the Weibull function. Moreover, predictions of average strength at different gauge lengths are also close to the experimental results, thus confirming that the converged solution properly accounts for the size effect in the range of lengths investigated (Table II). It may be emphasized that the Weibull parameters obtained for each length separately, have significant variability, and therefore, show poor confirmation with the Weibull size effect. Although the experiment is connected with variability of length only, one can expect the same effect for any volumetric difference.

4. While it is always risky to extrapolate the results of a statistical evaluation beyond the size range of the experimental data, the proposed procedure provides the opportunity to increase the number of simultaneously considered tests, thereby increasing the statistical reliability. Moreover, the approach can be used, in principle, for a sample in which each specimen has a unique length (size).

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